EXACT SOLUTION OF THE PROBLEM OF SUPERSONIC FLOW OF GAS PAST SOME THREE-DIMENSIONAL BODIES

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In papers [1] and [2] it was shown that a star-shaped cross-section of the body of a flying vehicle permits a ten-fold reduction in wave drag compared with the body of revolution of equivalent maximum cross-section area or volume. Since the deductions are based on the use of Newton's formula for the pressure distribution, it is necessary to discuss whether such drag reduction may be fictitious because of rise of the error in that formula. An answer could be found by comparing with exact solutions. However, for three-dimensional bodies only one exact solution is known [3], application of which is limited, as investigation has shown, to pyramidal bodies far in shape from the optimum. In view of this situation, it is of interest to construct an exact solution for three-dimensional star-shaped bodies, similar to those studied in [1].

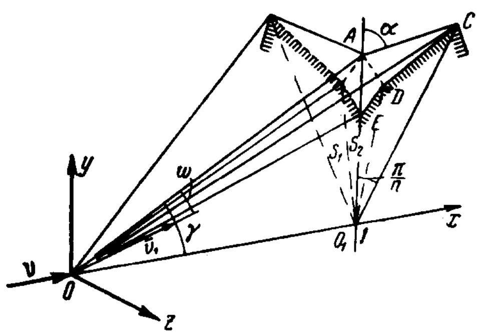

Fig. 1

We consider a system of plane intersecting shock waves passing through the origin of coordinates and determined by the angles α and γ (Fig.1). We denote the free-stream velocity and Mach number by U and M_∞, respectively. We introduce the auxiliary angle γ_1 according to Formula $\tan\gamma_1 = \tan\gamma\sin\alpha$, and then the velocity components behind the first shock wave, referred to the free-stream velocity, are given by the relations

$$v_x = 1-(1-\varepsilon)(\sin^2\gamma_1 - M_\infty^{-2}),\quad v_y = (1-\varepsilon)\cot\gamma_1(\sin^2\gamma_1 - M_\infty^{-2})\sin\alpha$$

$$v_z = -(1-\varepsilon)\cot\gamma_1(\sin^2\gamma_1 - M_\infty^{-2})\cos\alpha,\quad \varepsilon = (\varkappa-1)(\varkappa+1)^{-1}$$

$$(\varkappa \text{ is the adiabatic exponent}) \qquad (1)$$

Let the full velocity vector, referred to the velocity U and the Mach number behind the first shock wave be U_1 and M_1, respectively. The deflection of the flow after passing through the first shock wave (the angle δ) can be calculated from the relation

$$\tan\delta = \cot\gamma_1 \frac{(1-\varepsilon)(\sin^2\gamma_1 - M_\infty^{-2})}{1-(1-\varepsilon)(\sin^2\gamma_1 - M_\infty^{-2})}$$

After this M_1 is found from the well-known relations for an oblique shock wave. The disturbed gas flow behind a chosen system of shock waves will correspond to flow past a certain body if there exists a regular shock-wave intersection at point A, which permits the flow to be turned parallel to the plane of symmetry. The necessary condition for regular intersection is

$$M_{1n} = M_1 \sin\omega > 1 \tag{2}$$

The angle ω between the velocity vector U_1 and the line of intersection OA of the shock waves is found from the relation

$$\cos\omega\cos\gamma_1 = \cos(\gamma_1 - \delta)\cos\gamma \tag{3}$$

We now introduce into consideration the angle of inclination θ of the flow lying in the plane normal to the ridge OA and formed by the projections onto the plane of the segment O_1A and the velocity vector U_1. For its value we obtain, after some easy calculations, Formula

$$\cos\theta = (\tan\gamma - \tan\delta\sin\alpha)\,[\tan^2\gamma\,(1+\cos^2\alpha\tan^2\delta) + \tan^2\delta - 2\sin\alpha\tan\gamma\tan\delta]^{-1/2} \tag{3}$$

The position of the second wave (OAD in Fig.1) can now be determined by the angle β, lying in the plane of the angle θ, from the condition that the turning of the sream in this plane is given by Expression (3). As a result the angle β is found from the oblique shock wave equations in the form

$$\tan\theta = 2\cot\beta\,\frac{M_{1n}^2\sin^2\beta - 1}{M_{1n}^2(\varkappa + \cos 2\beta) + 2} \tag{4}$$

If the initial parameters of the problem M_∞, γ and α are such that Expression (4) has a solution for β, then the constructed gas flow corresponds to flow past a certain three-dimensional conical body with a cross section consisting of straight line segments. We determine its geometry (points E, D and C in Fig.1). After elementary calculations we find that points E and C have the coordinates

$$y_E = (\sin\alpha - \lambda\cos\alpha)\,\mathrm{tg}\,\delta, \qquad \lambda\,[\sin\gamma\cos\alpha\sin\delta - \cot(\beta-\theta)\cos\delta] = \cos\omega$$
$$y_c = z_c\cot(\pi/n), \qquad z_c\sin(\alpha - \pi/n) = \tan\gamma_1\sin(\pi/n) \tag{5}$$

To determine the remaining points it is necessary to know the coordintes of the point F lying on the continuation of the wall CD. A calculation leads to Expression

$$y_F = \frac{\tan\gamma_1\tan\delta\cos(\alpha-\pi/n)}{\sin(\pi/n)\tan\gamma_1 + \sin(\alpha-\pi/n)\cos\alpha\tan\delta} \tag{6}$$

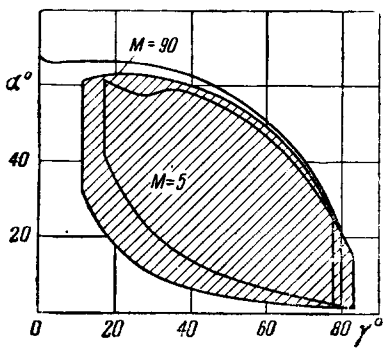

Fig. 2

Hence we see that the coordinates of point D can be found from Equations

$$y_D = \frac{y_F\cot(\beta-\theta) + \sin\gamma\cot(\alpha-\psi)}{\cot(\beta-\theta) + \cos\gamma\cot(\alpha-\psi)}$$
$$z_D = \frac{\sin\gamma - \cos\gamma\, y_F}{\cot(\beta-\theta) + \cos\gamma\cot(\alpha-\psi)} \tag{7}$$
$$\tan\psi = \left(1 - \frac{\tan\delta}{\tan\gamma_1}\right)\tan\left(\alpha - \frac{\pi}{n}\right)$$

As is evident from Fig.1, the projection of the region of disturbed flow behind the first and second shock waves onto the plane $x = 1$ is determined by the areas S_1 and S_2. Their values can be calculated by means of (5) to (7) from the relations

$$2S_1 = y_F(z_C - z_D), \qquad 2S_2 = y_E z_D$$

We proceed to the calculation of the forces. On the wall CD there acts a force produced by the constant pressure

$$c_p^{(1)} = 2(1-\varepsilon)(\sin^2\gamma_1 - M_\infty^{-2})$$

The wall ED lies in a region of increased pressure whose value is

$$c_p^{(2)} = c_p^{(1)} + 2U_1^2 \frac{\sin^2\omega(\sin^2\beta - M_{1n}^{-2})(1-\varepsilon)M_\infty^2\sin^2\gamma_1}{1+\varepsilon(M_\infty^2\sin^2\gamma_1 - 1)}$$

The flow in regions ACD and ADE (Fig.1) is uniform. All streamlines in the plane $x = 1$ passing through the shock wave AC converge at the single point E, which is a Ferri point for this flow. Then, upon traversing the second wave AD all streamlines, including the wall, have a corner. The wave drag of the body under consideration is represented by

$$C_x = \frac{c_p^{(1)}S_1 + c_p^{(2)}S_2}{S_1 + S_2}$$

A solution of the inverse problem exists only for a definite range of values of the parameters M_∞, α, γ and n. For example, the inequality $\alpha > \pi/n$ must always be satisfied. Obviously there are also other limitations. One such limitation is the necessary condition (2). Using it gives for each value of the free-stream Mach number a definite range of permissible values of the parameters α and γ. Fig.2 shows the results of the calculation for the range of Mach numbers $M_\infty = 5$, 10, and ∞ with the adiabatic exponent $\varkappa = 1.4$. The corresponding regions are cross-hatched in the figure. It mus be kept in mind that condition (2) is not sufficient, so that a choice of parameters inside the indicated regions has the character of a preliminary choice. Calculations of the dependence of wave drag upon the angle α for some values of the parameters M_∞, n and γ (with $\varkappa = 1.4$) are shown in Fig. 3 and 4.

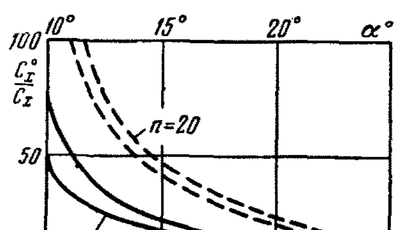

Fig. 3

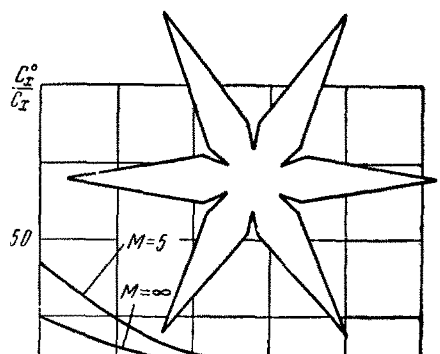

Fig. 4

The quantity C_x°/C_x indicated on the vertical axis represents the ratio of the drag of a circular cone of equivalent length and maximum cross-section area to the drag of the star-shaped body. Calculations for $M_\infty = \infty$ and $\gamma = 10°$ and 20° are shown in Fig.3. As is evident from the graph, the differences in drag decrease as the angle α increases. However, even for large values of α the drag of the star-shaped body still remains, say, ten times lower than that of the equivalent cone. As α decreases ($\alpha \to \pi/n$), the drag of the star-shaped body drops, and the difference from the cone increases. Comparison of the curves in Fig.3 shows that as the number n of edges increases, with the other parameters unchanged, the drag is reduced, whereas it increases as the parameter γ increases. Qualitatively similar behavior is found also for finite values of M_∞.

We point out one peculiarity of the curves given in Fig.4. According to the graph we find that as the M_∞ increases with the other parameters unchanged, the drag coefficient of the body increases. In fact, different bodies are obtained for different Mach numbers, and the change in geometry of the body affects the drag more strongly than variation of Mach number. Fig.4 also shows the shape of the cross section for one case with the values of the parameters $n = 6$, $\gamma = 5°$, $\alpha = 41°$, and $M_\infty = \infty$. Together with the exact calculation of wave drag, the calculation was also carried out according to Newton's formula for a number of cases. Comparison shows that the error has the order of magnitude of 20 per cent. Thus all results on significant changes of wave drag obtained for star-shaped bodies in previous works [1 and 2] agree quantitatively as well as qualitatively with the exact solutions.

We observe in conclusion that the possibility of similar solutions was conjectured independently by Maikapar [4].

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